Christ Church Grammar School

2021
TEST 2

MATHEMATICS SPECIALIST Year 12
Section One:
Calculator-free

Your name SOLUTIONS

Teacher's name ____________________

Time and marks available for this section

Reading time for this section: 2 minutes
Working time for this section: 25 minutes
Marks available: 25 marks

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

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3. Answer all questions.

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6. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

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Question 1 **(7 marks)**

Functions f and g are defined such that: $f(x) = \sqrt{4-x}$ $g(x) = \frac{2}{x-2}$

(a) Determine $gf(x)$. (1 mark)

$gf(x) = \frac{2}{\sqrt{4-x}-2}$ ✓

(b) Determine the domain and range for $gf(x)$. (4 marks)

$D_{gf(x)} = \{ x \in \mathbb{R} : x \leq 4 \cap x \neq 0 \}$ ✓ ✓

1 correct inequality
2nd correct inequality with intersection (can use and)

$R_{gf(x)} = \{ y \in \mathbb{R} : y > 0 \cup y \leq -1 \}$ ✓ ✓

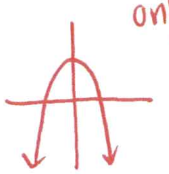
1 correct inequality
2nd correct inequality with union.

$f(x) \Rightarrow D_{f(x)}\ \{x \leq 4\}$ $g(x) \Rightarrow D_{g(x)}\ \{x \neq 2\}$.

$R_{f(x)}\ \{y \geq 0\}$ $R_{g(x)} = \{y \neq 0\}$

(c) Given that $f^{-1}(x) = 4 - x^2$, explain if it is true that $f^{-1}(-1) = 3$. (2 marks)

$f^{-1}(x) = 4 - x^2$ only if $x \geq 0$

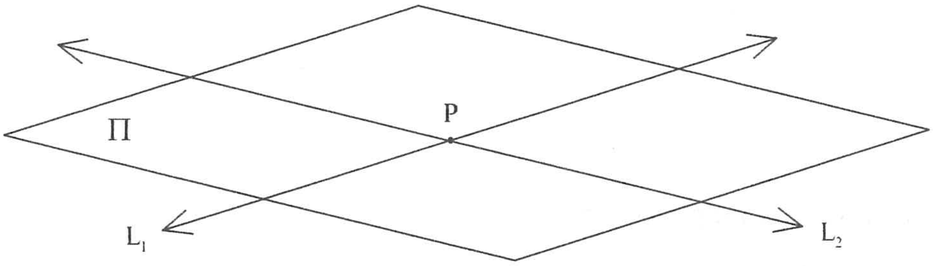


No, ✓ it is not true as $f^{-1}(x)$ does not exist when $x < 0$. ✓

answer
valid reason.

Question 2 **(9 marks)**

The lines L_1 and L_2 have equations $\boldsymbol{r} = (3+\lambda)\boldsymbol{i} + (1+\lambda)\boldsymbol{j} - 2\lambda\boldsymbol{k}$ and $\boldsymbol{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \mu\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$ respectively, with $a, b, c \in \mathbb{R}$, and they lie on the same plane Π as shown.



(a) Given that the lines intersect at the point P when $\lambda = 3 = \mu + 2$, determine the value of the constants a, b and c, and the exact distance of point P from the origin. (4 marks)

$$r_{L_1} = \begin{pmatrix} 3+\lambda \\ 1+\lambda \\ -2\lambda \end{pmatrix} \quad r_{L_2} = \begin{pmatrix} a+4\mu \\ b-2\mu \\ c+\mu \end{pmatrix}.$$

when $\lambda = 3$ $r_{L_1} = \begin{pmatrix} 6 \\ 4 \\ -6 \end{pmatrix}$ and $\mu = 1$ $(3-2)$ $r_{L_2} = \begin{pmatrix} a+4 \\ b-2 \\ c+1 \end{pmatrix}$

$\therefore$ $6 = a+4 \Rightarrow a = 2$ ✓

$4 = b-2 \Rightarrow b = 6$ ✓

$-6 = c+1 \Rightarrow c = -7$ ✓

equates L1 + L2 and correctly solves for one constant

correctly solves for 2nd + 3rd constant.

$\overrightarrow{OP} = \begin{pmatrix} 6 \\ 4 \\ -6 \end{pmatrix}$ ✓ $\therefore |OP| = \sqrt{6^2+4^2+(-6)^2}$

$= \sqrt{88}$

$= 2\sqrt{22}$ units. ✓

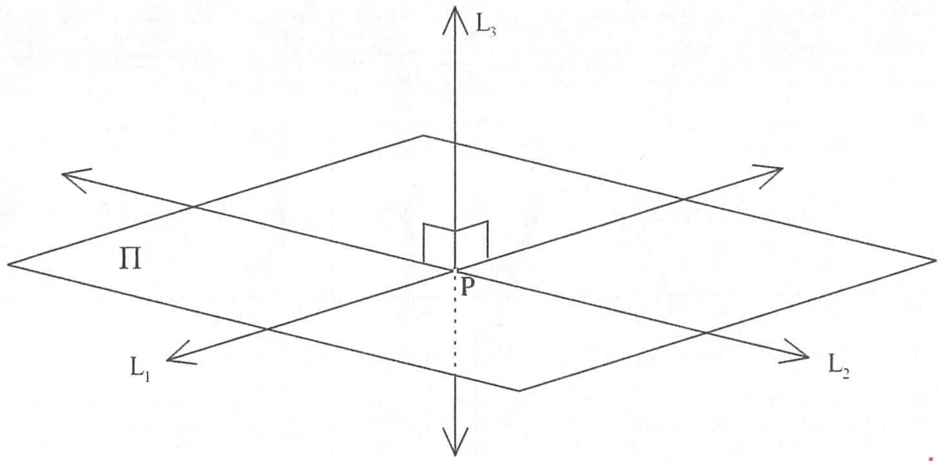
identifies vector from P to origin

correctly calculates exact distance.

(Note: this can be based on their vector if clearly working shown).

Question 2 continued

(b) A third line L_3 is perpendicular to the plane formed by L_1 and L_2, and passes through P. Determine the vector equation of L_3 and the Cartesian equation of the plane Π. (5 marks)



$$n = L_1 \times L_2 = \begin{pmatrix}1\\1\\-2\end{pmatrix} \times \begin{pmatrix}4\\-2\\1\end{pmatrix} = \begin{pmatrix}1\times1-(-2)(-2)\\-(1\times1-4(-2))\\1\times(-2)-4\times1\end{pmatrix} = \begin{pmatrix}-3\\-9\\-6\end{pmatrix} \text{ or } -3\begin{pmatrix}1\\3\\2\end{pmatrix}$$

✓ correctly sets up cross product

✓ correctly determines crossproduct or another // to $\begin{pmatrix}1\\3\\2\end{pmatrix}$.

$$L_3 = \begin{pmatrix}6\\4\\-6\end{pmatrix} + \beta\begin{pmatrix}1\\3\\2\end{pmatrix}$$

✓ states normal vector (implicitly in a vector eqn) or explicitly as $\underset{\sim}{n} = \begin{pmatrix}1\\3\\2\end{pmatrix}$

$$\underset{\sim}{n}.\underset{\sim}{r} = \underset{\sim}{a}.\underset{\sim}{n}$$

$$\underset{\sim}{r}.\begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}6\\4\\-6\end{pmatrix}.\begin{pmatrix}1\\3\\2\end{pmatrix}$$

$$\underset{\sim}{r}\begin{pmatrix}1\\3\\2\end{pmatrix} = 6+12-12$$

$$\underset{\sim}{r}.\begin{pmatrix}1\\3\\2\end{pmatrix} = 6$$

✓ determines vector eqn as the normal form

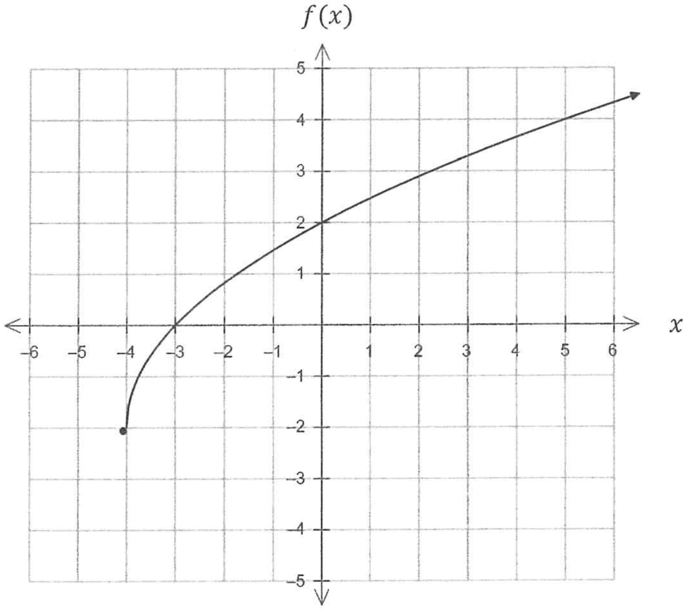
$$\therefore \; x + 3y + 2z = 6 \; ✓$$

determines Cartesian eqn (based on their $\underset{\sim}{n}$ and k value)

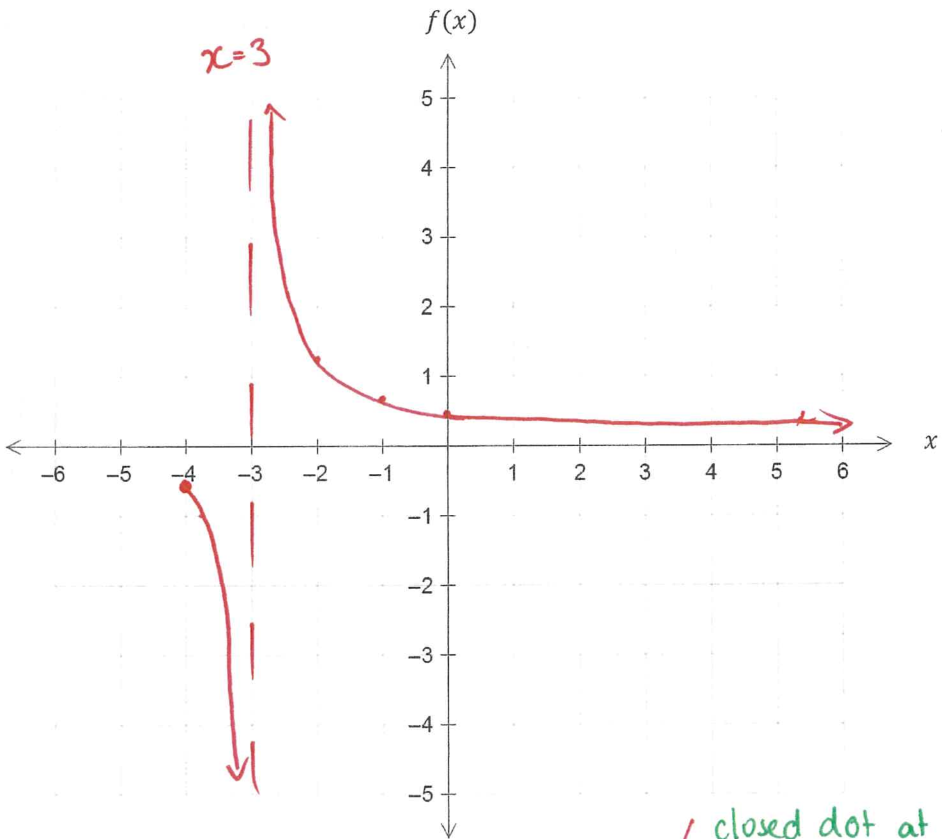
Note: This is based on their cross product.

Question 3 **(9 marks)**

The graph of $f(x) = 2\sqrt{x+4} - 2$ is shown below.



(a) Sketch the graph of $\frac{1}{f(x)}$ on the grid below. (3 marks)



✓ closed dot at $(-4, -\frac{1}{2})$
smooth curve, as $x \to -3^-$, $y \to -\infty$

✓ vertical asymptote at $x = -3$
horizontal asymptote at $y = 0$

✓ smooth curve, $x \to -3^+$, $y \to \infty$
and $x \to \infty$, $y \to 0^+$.

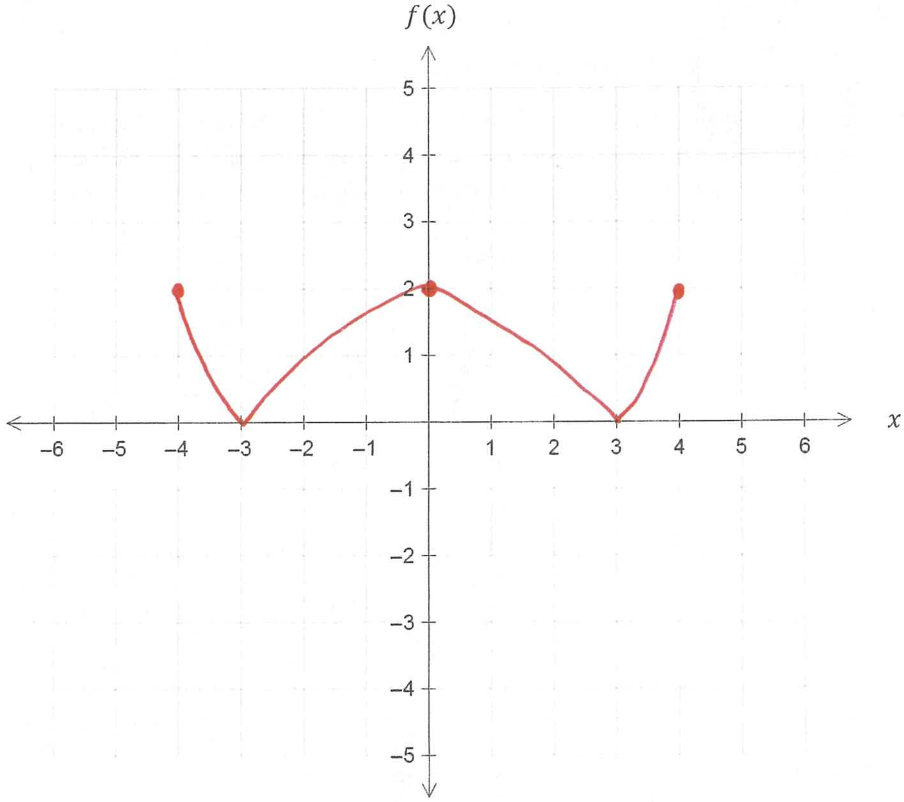
Question 3 continued

(b) Sketch the graph of $|f(-|x|)|$ on the grid below. (3 marks)

✓ roots at $x = \pm 3$ and y-int. at $y = 2$

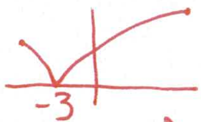
✓ symmetry of graph

✓ correct curvature



(c) The domain of $f(x)$ is restricted to $x \geq k$ so that the inverse of $g(x) = |f(x)|$ exists. Determine the value of k and state the domain and range of $g^{-1}(x)$. (3 marks)

$f(x) \Rightarrow D_x = \{x \in \mathbb{R} : x \geq -4\}$

$|f(x)| =$ 

$= g(x)$

$f(x) = 2\sqrt{x+4} - 2$

$D_{g(x)} \Rightarrow x \geq -3$

$R_{g(x)} \Rightarrow y \geq 0$

$\therefore k = -3$ ✓ states correct 'k' value

$D_{g^{-1}(x)} = \{x \in \mathbb{R} : x \geq 0\}$ ✓ states correct domain with correct notation

$R_{g^{-1}(x)} = \{y \in \mathbb{R} : y \geq -3\}$ ✓ states correct range with correct notation.

Note: only penalise once if correct notation not used.

End of questions

Additional working space

Question number: _________

Additional working space

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TEST 2

MATHEMATICS SPECIALIST Year 12
Section Two:
Calculator-assumed

Your name SOLUTIONS.

Teacher's name ____________________

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Working time for this section: 20 minutes
Marks available: 21 marks

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Special items: drawing instruments, templates, and up to three calculators approved for use in this assessment

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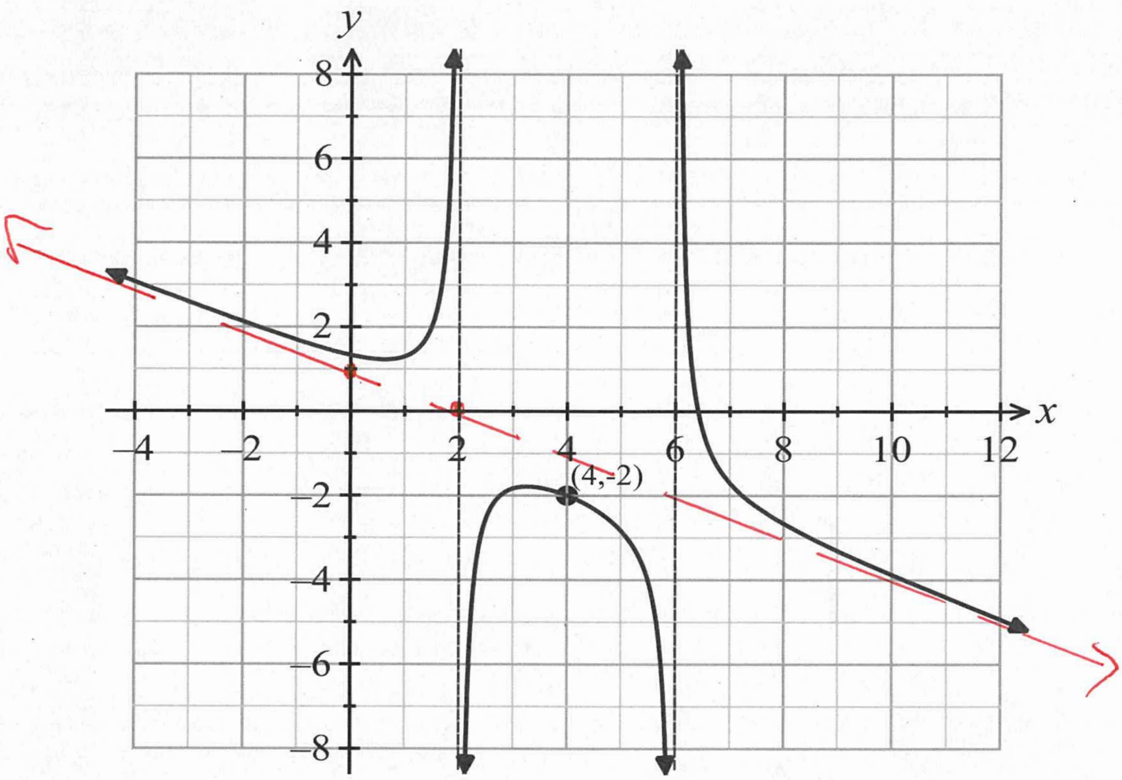
Question 4 **(5 marks)**

The function $f(x) = ax + b + \frac{k}{(x+c)(x+d)}$ is shown below, where $a, b, c, d, k \in \mathbb{R}$.



State the value of the constants a, b, c, d and k.

VA: $x = 2$, $x = 6$ $\therefore$ $c = -2$, $d = -6$ ✓ ✓ accept (~~or~~ $c = -6$, $d = -2$)

OA: $y = -\frac{1}{2}x + 1$ $\therefore$ $a = -\frac{1}{2}$ (or -0.5) ✓

$b = 1$ ✓

When $x = 4$, $y = -2$ $\therefore$ $-2 = -\frac{1}{2}(4) + 1 + \frac{k}{(4-2)(4-6)}$

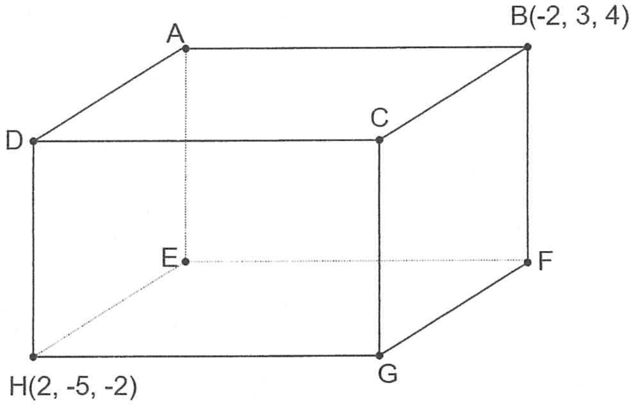
$\therefore k = 4$ ✓

Note: can gain follow through marks for 'k' if 'c' or 'd' ~~or~~ 'a' or 'b' are incorrect and they show working.

Question 5 **(6 marks)**

The right rectangular prism ABCDEFGH shown is positioned in the Cartesian coordinate system such that AD is parallel to the x-axis, AB is parallel to the y-axis, and EA is parallel to the z-axis.

The vertices B and H have coordinates $(-2, 3, 4)$ and $(2, -5, -2)$ respectively.



(a) State the coordinates of vertex E. (1 mark)

$E(-2, -5, -2)$ ✓

(b) Determine the percentage of the prism that lies in the first octant. (2 marks)

$\frac{V_o}{V_T} \times 100$

$= \frac{2 \times 3 \times 4}{4 \times 8 \times 6} \times 100$ ✓

$= 12.5\%$ ✓

↓ all positive 'x', 'y', 'z' values.

Note: correct answer; no working is 2 marks.

Question 5 continued

(c) Determine the vector equation of the sphere that has HB as its diameter.

(3 marks)

midpoint of HB = $(0,-1,1)$ ✓

states the midpoint of HB as this is centre of the sphere.

diameter $= |HB| = \sqrt{4^2+8^2+6^2}$

$= \sqrt{116} = 2\sqrt{29}$

$\therefore$ radius $= \frac{1}{2}|HB|$

$= \sqrt{29}$ ✓

calculates the radius of the sphere

$\therefore \left| \underset{\sim}{r} - \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right| = \sqrt{29}$ ✓

writes vector equation

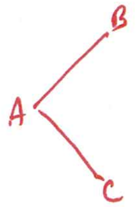
also accept

$\left| \underset{\sim}{r} + \underset{\sim}{j} - \underset{\sim}{k} \right| = \sqrt{29}$

Question 6 **(10 marks)**

Triangle ABC in space has vertices with position vectors $\overrightarrow{OA} = \boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$, $\overrightarrow{OB} = -2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ and $\overrightarrow{OC} = 2\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$.

(a) Determine the size of $\angle BAC$ correct to the nearest degree. (3 marks)



$$\angle BAC = \cos^{-1}\left(\frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}|\,|\overrightarrow{AC}|}\right) \qquad \overrightarrow{AB} = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$$

✓ determines vectors $\overrightarrow{AB}$ & $\overrightarrow{AC}$

$$= \cos^{-1}\left(\frac{-3 + 15 + 18}{\sqrt{27} \times \sqrt{62}}\right) \qquad \overrightarrow{AC} = \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$$

✓ determines magnitudes of their vectors $|\overrightarrow{AB}|$ & $|\overrightarrow{AC}|$.

$$= 42.84^\circ$$

$$\simeq 43^\circ$$

✓ correct value or states angle based on their vectors if not correct answer. Must see evidence

(b) Find the vector equation $\boldsymbol{r} = \boldsymbol{p} + \lambda \boldsymbol{d_1} + \mu \boldsymbol{d_2}$ of the plane Π that contains triangle ABC. (1 mark)

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$$ ✓

or $$\underset{\sim}{r} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$$ or other multiples of direction.

(c) The line L has equation $\boldsymbol{r} = \alpha \begin{pmatrix} m \\ 7 \\ n \end{pmatrix}$ and it is perpendicular to the plane Π found in part (b).

(i) Show that $m = 1$ and $n = -6$. (2 marks)

✓ determines cross product correctly

$$\begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 18 - 15 \\ -(-18 - 3) \\ -15 - 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 21 \\ -18 \end{pmatrix} \Rightarrow 3 \begin{pmatrix} 1 \\ 7 \\ -6 \end{pmatrix}$$

$\therefore m = 1$

$n = -6$.

✓ shows $\alpha \begin{pmatrix} m \\ 7 \\ n \end{pmatrix}$ to state m and n.

Note: must show some evidence of correct cross product to get full marks.

Question 6 continued

(ii) Find the point of intersection between the line L and the plane Π.

(3 marks)

For plane Π $\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$ $\underset{\sim}{n} = \begin{pmatrix} 1 \\ 7 \\ -6 \end{pmatrix}$ $\underset{\sim}{a} \cdot \underset{\sim}{n} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 7 \\ -6 \end{pmatrix}$

So $\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 7 \\ -6 \end{pmatrix} = -7$ ✓ determines vector eqn of plane Π $= -7$

Subst $\underset{\sim}{r} = \alpha \begin{pmatrix} 1 \\ 7 \\ -6 \end{pmatrix}$ into $\underset{\sim}{r}$ for plane Π

So $\alpha \begin{pmatrix} 1 \\ 7 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 7 \\ -6 \end{pmatrix} = -7$

$86\alpha = -7$

$\alpha = \frac{-7}{86}$ ✓ solves for α

states correct co-ordinate point P. ✓

$\overrightarrow{OP} \Rightarrow \underset{\sim}{r} = \frac{-7}{86}\begin{pmatrix} 1 \\ 7 \\ -6 \end{pmatrix}$ $\therefore \overrightarrow{OP} = \begin{pmatrix} -7/86 \\ -49/86 \\ 42/86 \end{pmatrix}$ or $\frac{1}{86}\begin{pmatrix} -7 \\ -49 \\ 42 \end{pmatrix}$ etc $\therefore P = \left(\frac{-7}{86}, \frac{-49}{86}, \frac{42}{86}\right)$

(d) Determine the shortest distance of the plane Π from the origin. (1 mark)

L is $\perp$ to plane and they intersect at P

$\therefore$ shortest distance is $|\overrightarrow{OP}|$

$\therefore |\overrightarrow{OP}| = \frac{7}{86}\sqrt{(-7)^2+(-7)^2+(6)^2}$

$= \frac{7}{86}\sqrt{86}$ ✓

accept either answer.

or $\simeq 0.75$ units ✓

End of questions

Additional working space

Question number: _________

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